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Existence of a Constant for Finite System Extinction

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We show the existence of a constant $\gamma \in (0, \infty)$ such that if τ^n is the extinction time for a supercritical contact process on $[1, n]^d$ starting from full occupancy, then $\{\log(E[\tau^n])\}/n^d$ tend to γ as n tends to infinity.

KEY WORDS: Finite particle systems; contact process.

INTRODUCTION

We consider the contact process restricted to D, a finite subset of Z^d. That is to say the contact process where no births (or occupied sites) are permitted outside D, but which otherwise evolves as a contact process. Formally our process is a continuous time Markov chain on the (finite) state space $\{0, 1\}^D$ with generator

$$\Omega f(\xi) = \sum_{x \in D} (f(\xi^x) - f(\xi))\, c(x, \xi)$$

where $c(x, \xi) = 1$ if $\xi(x) = 1$; $= \lambda \sum_{y \in D,\, |x-y|_1 = 1} \xi(y)$ if $\xi(x) = 0$, and $\xi^x(z) = \xi(z)$ for $z \neq x$ and $\xi^x(x) = 1 - \xi(x)$.

This process is a continuous time Markov chain on $\{0, 1\}^D$ for which $\underline{0}$ (the configuration ξ so that $\xi(y) = 0\ \forall y$) is a trap state but otherwise all states can be reached from all other states. Thus this process must eventually hit the trap state $\underline{0}$ and thereafter stay there. The question we pursue in this paper is how, starting from all sites having value 1, the time to hit the trap state behaves for the set $D = [1, n]^d$ as n becomes large.

We will write ξ_t^D for the contact process restricted to D at time t. Unless otherwise specified it will be assumed that these processes are all starting from full occupancy on D and that they are all generated from the

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same Harris system of Poisson processes on Z^d. See [L] or [D] for further details on the Harris construction of the contact process. It follows from the above assumptions and the attractiveness of the processes that $\xi_t^{D_1} \leqslant \xi_t^{D_2}$ for all t if $D_1 \subseteq D_2$. A configuration on $\{0, 1\}^D$ may be thought of as either a function from D to $\{0, 1\}$ or as subset of D (by identifying ξ^D with the subset of sites taking value 1). So we may also write the above relation as $\xi^{D_1} \subseteq \xi_t^{D_2}$ for all t if $D_1 \subseteq D_2$.

In the interests of avoiding overcrowded suffixes for the special cases $D = \{0, 1\}^{[1, n]^d}$, we will write ξ_t^n rather than the more consistent $\xi_t^{[1, n]^d}$. Similarly, while we use τ^D to denote the hitting time of for ξ^D, we will denote the random hitting time of $\underline{0}$ for ξ^n by τ^n. As mentioned above the default value of ξ_0^D will be simply D, the process $\xi_t^{D, x}$ will denote the process with $\xi_0^{D, x}$ equal to $\{x\}$. Again it will be assumed that this process is generated by the fixed Harris system and thus we will have (again using the attractiveness of the systems) for all t, $\xi_t^{D, x} \subset \xi_t^D$.

Our note aims to prove

Theorem 0.1. There exists a constant $\gamma \in (0, \infty)$ so that for supercritical contact process

$$\frac{\log(E[\tau^n])}{n^d} \to \gamma$$

as n tends to infinity.

In our proof we will explicitly treat the case $d = 2$ but it will be clear that the arguments require no extra thoughts for higher dimensions just more notation. Throughout the paper terms like K will denote constants which do not depend on variables such as n and may change from line to line.

Remark. In fact it will be clear, though we will not prove this, that for any bounded connected set $D \subset R^d$ whose indicator function is Riemann integrable (that is as n tends to infinity, the number of nth order dyadic cubes that intersect D but are not contained in D is negligible compared to the number of such cubes that are completely contained in D) if $D_n = nD \cap Z^d$ then

$$\frac{\log(E[\tau^{D_n}])}{n^d} \to \gamma vol(D)$$

as n tends to infinity.

Our problem follows work by [DS] for the contact process in one dimension. There Theorem 0.1 was shown and the constant γ explicitly identified. This followed work by [DL] which showed that the behaviour of extinction times for large finite systems reflected the phase transition occurring at the critical rate λ_c. This behaviour can be seen to carry over to higher dimensions. [C] showed that in the supercritical regime there exists two constants C_1, C_2 so that for large $ne^{C_1 n^d} \leqslant E[\tau^n] \leqslant e^{C_2 n^d}$. We shall use the breakthrough renormalization result of [BG] together with contour arguments found in or based on [D] to prove our result. This combination of ideas has been used before in [M] to show that τ^n divided by its expectation converged in distribution to the mean one exponential random variable. This note has much in common with the earlier work.

The basic step to proving Theorem 0.1 is to show

Corollary 3.1. Let $\tau^{2n,n}$ be the time for the contact process restricted to $[1, 2n] \times [1, n]$ to die. For all n large, $E[\tau^{2n,n}] \geqslant n^{-9} E[\tau^n]^2$.

Given this result the final conclusion follows from routine subadditive manipulations. The key part of the proof of Corollary 3.1 is the introduction of an artificial Markov process which resembles the contact process on $\{0, 1\}^{[1,n]^2}$ but which has a regeneration property that turns it into an irreducible Markov process with a stationary measure which puts mass on $\underline{0}$ "approximately" equal to $1/E[\tau^n]$. "Approximately" here means up to a multiple of a power of n. If we run two such independent processes, then the corresponding stationary measure for the double configuration $(\underline{0}, \underline{0})$ will be "approximately" $1/(E[\tau^n])^2$. By the attractiveness property (and our not entirely straightforward choice of regeneration procedure) it will follow that we can couple these two independent processes with the contact process on $[1, n] \times [1, 2n]$ so that (outside of an irrelevantly small set) our contact process will be null only if the two independent processes are in state $(\underline{0}, \underline{0})$. From this Corollary 3.1 will be argued.

1. In this section we gather together some of the consequences of [BG] and the contour arguments of [D1]. We assume familiarity with oriented bond percolation and the arguments found in the latter reference.

Lemma 1.1. For any fixed $\lambda > 0$, there is an $\varepsilon_0 > 0$ so that if $\psi(s, t)$ is a 1-dependent oriented site percolation system restricted to $[1, n] \times Z_+$ with probability at least $1 - \varepsilon_0$ of a given site being open, then the chance that $(x, 0)$ is connected to (y, t) for $x, y + t \equiv 0 \pmod 2$, $t \in [n^2 \lambda, n^8]$ is at least 1/2, uniformly over x, y, t subject to above conditions.

In the following as is usual $B(x, r)$ denotes the Euclidean ball of radius r centred at x and $\psi_m^A = \{z: (z, m) \text{ is connected to } (x, 0) \text{ for some } x \in A\}$.

[BG] established a powerful renormalization result. For two dimensions the renormalization result entails:

Theorem 1.1. Given a supercritical contact process ξ and any $\varepsilon > 0$, there exists integers r, $L(r \ll L)$, S, k so that we can couple ξ restricted to $Z \times [0, 4L]$, with a 1-dependent percolation system, ψ, of closure probability $<\varepsilon$, so that for $A = \{x: x$ is even and $\xi_0 \equiv 1$ on $B(y, r)$ for some $y \in [xkL - 2L, xkL + 2L] \times [L, 3L]\}$, $\psi^A_m \subset \{x: x + m \equiv 0 \pmod 2)$, $\xi_s \equiv 1$ on $B(y, r)$ for $y \in [xkL - 2L, xkL + 2L] \times [L, 3L]$ and some $s \in [2mS, 2mS + 2S]\}$

Putting these two results together we have

Corollary 1.1. For any fixed $\lambda > 0$ there exists L and $c > 0$ such that for the contact process restricted to $D_n = [1, n] \times [1, 4L]$, ξ^{D_n},

$$\inf_{\lambda n^2 \leqslant t \leqslant n^8} \inf_{x, y \in D_n} P[\xi^{D_n, x}_t(x) = 1] \geqslant c > 0$$

for all n large.

Proof. Let us choose L so that Theorem 1.1 can be applied with L and some k, r, S for the ε_0 of Lemma 1.1. Then given x, choose z so that z is even, $zkL \in (2L, n - 2L]$ and subject to this $(zkL, 2L)$ minimizes the distance from x. Necessarily this distance will be less than $5kL + 4L$. Now choose w similarly so that w is even and $(wkL, 2L)$ minimizes the distance to y. Then choose even u as large as possible subject to $2(k+1)\, uS$ being less than $t - 2$. Let B_1 be the event that $B((zkL, 2L), r)$ is fully occupied by $\xi^{D_n, x}_1$. $P[B_1] \geqslant a(r, k, L) > 0$. By Theorem 1.1, given event B_1, the chance of B_2, that for some v in D_n within $4L$ of $(wkL, 2L)$ and for some s in $[2ukS, 2(k+1)\, uS]$, we have $\xi_s(v) = 1$ is at least 1/2. Using the Strong Markov property at

$$T = \inf\{s \in [2ukS, 2u(k+1)\, S]: \xi^{D_n, x}(v) = 1 \text{ for some } |v - (wkL, 2L)| \leqslant 4L\}$$

we obtain that $P[\xi^{D_n, x}_t(y) = 1 \mid B_1 \cap B_2] \geqslant b(k, L) > 0$. Thus our lemma holds with c equal to $\frac{1}{2} a(r, k, L)\, b(k, L)$.

Corollary 1.2. For c the constant of Corollary 1.1, we have that given finite M, $\lambda > 0$, for all n large if $D_n = [1, n] \times [1, Mn]$, then

$$\inf_{\lambda n^2 \leqslant t \leqslant n^8} \inf_{x, y \in D_n} P[\xi^{D_n, x}_t(y) = 1] \geqslant c^2 > 0$$

We will also need the following lemma from [M].

Lemma 1.2. For n large and all $x \in [1, n]^2$, $P[\xi^{1,n}_{n^2} = \xi^{x,n}_{n^2} \mid \xi^{x,n}_{n^2} \neq \underline{0}] > \frac{1}{2}$.

Remark. The above conclusion also holds for domains $[1, n] \times [1, Mn]$ for M fixed and all large n.

2. In this section we introduce a family of Markov processes which are essentially Markov chains on $\{0, 1\}^{[1, n]^d}$ that greatly resemble the contact process but which have no trap states. Our process will be denoted by (η_t, v_t) with $\eta_t \in \{0, 1\}^{[1, n]^d}$ and $v_t \in [0, n^8)$. So our process (η_t, v_t) has state space $\{0, 1\}^{[1, n]^d} \times [0, n^8)$, though since we will stipulate that if $\eta_t \neq \underline{0}$ then v_t must equal 0, the state space may also be seen as being $\{0, 1\}^{[1, n]^d} \cup \underline{0} \times [0, n^8)$.

The process v_t will be the measure (mod(n^8) of how long η has been in state $\underline{0}$. The choice of the power 8 is somewhat arbitrary; any large power would do. It corresponds to the fact that (for the supercritical contact process in a thick enough region) if a site x is vacant for the contact-process but a site within of order n is occupied then the chance that site x is occupied at time (say) n^2 will be bounded away from 0 as n becomes large. Hence if a fixed site x is vacant throughout time interval $[0, n^8/2]$ then outside an event of probability even $e^{-cn^4} \ll e^{-kn^2}$ for any k, there must have been a time in interval $[0, n^8/2)$ at which there were no sites occupied for the contact process within of order n of our site x.

Fix z in $[1, n]^d$ and $\delta \in (0, 1)$, then our process can be generated by a Harris system on $\{0, 1\}^{[1, n]^d}$ and an independent system of i.i.d. Bernoulli (δ) random variables X_i. The process is defined by alternating epochs: a type one epoch is where η is not identically zero (and therefore v is 0), while a type two epoch is where v is non-zero and therefore η is identically zero. During a type 2 epoch v simply linearly increases (mod(n^8)) until it hits value zero. At this point with probability $1 - \delta$ the type two epoch (and the linear increase of v) continues and with probability δ, the type two epoch ends and η jumps to configuration η^z where $\eta^z(y) = \delta_{xy}$, the Kronecker function for z. Into type one epoch v remains zero while η evolves like a contact process powered by the Harris system. The one epoch ends and a two epoch begins when η again hits the empty configuration.

We record some simple properties of this Markov process.

Lemma 2.1. Let π^n be the projection of the equilibrium distribution of (η^n, v^n) on $\{0, 1\}^{[1, n]^d}$. Then $\pi^n\{\underline{0}\} \leqslant Kn^8/E[\tau^n]$ for some constant K not depending on n.

Proof. For any initial state $\pi^n\{\underline{0}\}$ is the a.s. limit of

$$\frac{1}{t}\int_0^t I_{\eta_s^n=\underline{0}}\, ds$$

We will consider the process starting from $(\underline{0}, 0)$. Then in this process the successive type one and two epochs form an alternating renewal sequence. If Y_i is the length of the ith type two epoch and Z_i is the length of the ith type one epoch, then the Y_i and the Z_i are i.i.d. sequences and $\pi^n\{\underline{0}\}$ is equal to $E[Y_1]/(E[Y_1]+E[Z_1]) \leqslant E[Y_1]/E[Z_1]$. Now $E[Y_1]$ is equal to n^8/δ. We cannot calculate $E[Z_i]$ exactly however we can bound from below. By Lemma 1.2, there exists a constant $c>0$ independent of n or z in $[1, n]^d$ such that with probability at least c, $\xi_{n^2}^{z,n} = \xi_{n^2}^{1,n}$. Now by [M], the lifetime τ satisfies $\tau/E[\tau]$ tends in distribution to a mean one exponential, so for any sequence of events $A(n)$ of probability at least c we have

$$\liminf_{n\to\infty} \frac{E[\tau^n; A(n)]}{E[\tau^n]} \int_0^{-\log(1-c)} ve^{-u}\, du \geqslant 2K$$

for some K not depending on n. Therefore

$$\frac{E[\tau^n; I_{\{\xi_{n^2}^{x,n} = \xi_{n^2}^n\}}]}{E[\tau^n]} \geqslant K$$

for n large. Therefore $E[Z_1] \geqslant KE[\tau^n]$ for all n large.

Corollary 2.1. Let (η^n, v^n) be a process in equilibrium, then the probability that η is vacant for some $t \in (0, n^8)$ is bounded by $Kn^8/E[\tau^n]$ for n large.

Proof. Let X be the Lebesgue measure of the time in $(0, 2n^8)$ that η is vacant. Then as (η, v) is in equilibrium, the expectation of X is equal to $2n^8\pi^n(\underline{0}) \leqslant 2n^8\, Kn^8/E[\tau^n]$. Let S be the first time that η becomes zero, then by the Strong Markov property we have $E[X \mid S \leqslant n^8] \geqslant n^8$ as after hitting $\underline{0}\eta$ must remain there for at least n^8 time units. Hence we must have that $P[S < n^8] \leqslant Kn^8/E[\tau^n]$.

Corollary 2.2. Let (η^n, v^n) be a process in equilibrium, and let $(\eta^{n,1}, v^n)$ be the process started with $\eta_0^{n,1} \equiv 1$ and generated by the same Poisson processes and the same Bernoulli random variables. With probability at least $1 - Kn^8/E[\tau^n]$ (for some K not depending on n), we have $n_s^n \leqslant \eta_s^{n,1}\ \forall s$.

Proof. It is easy to see that

$$P(\eta_s^{n,1} \equiv 0 \text{ for some } s \in (0, n^8)) \leqslant \frac{Kn^8}{E[\tau^n]}$$

for some K not depending on n by [M] and attractiveness. By Lemma 1.2 repeatedly applied we have

$$P[\eta_{n^8}^n \neq \eta_{n^8}^{n,1}, \eta_s^n \neq \underline{0}, \eta_s^{n,1} \neq \underline{0}\ \forall s \in [0, n^8]] \leqslant e^{-kn^4}$$

for some strictly positive k. The result now follows from Corollary 2.1.

Remark. If in the above lemmas, the area to which the contact process is restricted is changed from $[1, n]^2$ to $[1, n] \times [1, Mn]$ for some fixed positive M, then all the above results and arguments go through. If we have a fixed n and $n/M \leqslant m \leqslant Mn$, then we denote the corresponding process on $\{0, 1\}^{[1,n] \times [1,m]} \cup \underline{0} \times [0, n^8)$ by $(\eta^{n,m}, v^{n,m})$. If M is fixed then as n tends to infinity, the fact that the recovery time is n^8 rather than m^8 or some term combining n and m becomes mute as the specific recovery time is irrelevant. For the rest of the paper $\tau^{n,m}$ will represent the time for $\xi^{n,m} = \xi^{[1,n] \times [1,m]}$ to become vacant.

3. In this section we introduce a coupling between two independent processes (η_t, v_t), (η'_t, v'_t) on $\{0, 1\}^{[1,n]^2}$ and the contact process restricted to $[1, 2n] \times [1, n]$, $\xi^{2n,n}$, provided that these constant δ used in the definition of the (η_t, v_t) has been fixed sufficiently small.

Proposition 3.1. For fixed finite M let $n/M \leqslant m, m' \leqslant Mn$. We can couple together two independent processes $(\eta_t^{m,n}, v_t^{m,n})$, $(\eta_t^{m',n}, v_t^{m',n})$ in equilibrium, with a contact process restricted to $[1, m+m'] \times [1, n]$, $\xi^{m+m',n}$ so that until the random time $V_n = \inf\{t: \eta_t = \eta'_t = \underline{0}\}$, we have $\xi_t^{m+m',n} \neq \underline{0}$, outside of a set of probability e^{-cn^4}.

Proof. We simply treat the case where $n = m = m'$ for the sake of simplicity. Fix z in $[1, n]^2$ and a $\delta < c^2/2$, where c is the constant of Corollary 1.1. These choices will define the distribution of the processes (η, v), (η', v') as in Section 2. Let $\xi^{2n,n}$ be generated by Poisson processes D_x for $x \in [1, 2n] \times [1, n]$, $N_{x,y}$, for $x, y \in [1, 2n] \times [1, n]$, $|x - y| = 1$. We will use these same processes to generate the other two processes: to generate (η, v) we use the Poisson processes D_x, $N_{x,y}$ for $x, y \in [1, n]^2$; to generate (η', v') we use the Poisson processes N', D' given by $D'_x = D_{x+(n,0)}$, $N'_{x,y} = N_{x+(n,0),\ y+(n,0)}$. It remains to describe the Bernoulli (b) random variables to complete the coupling.

We define the sequence of stopping times T_i, S_i recursively as follows (no matter that we have not yet fully constructed the (η, v), (η', v') processes): $T_0 = 0$, $S_i = \inf\{t > T_{i-1} : \eta_t$ or $\eta'_t = \underline{0}\}$, and $T_i = S_i + n^8$. By our choice of generating Poisson processes, it is clear that if

$$\xi^{2n,n}_{T_i}(x) \geqslant \eta_{T_i}(x)\ \forall x \in [1, n]^2 \qquad \text{and}$$
$$\xi^{2n,n}_{T_i}(x) \geqslant \eta'_{T_i}(x - (n, 0))\ \forall x \in [n+1, 2n] \times [1, n]$$

then

$$\xi^{2n,n}_{t}(x) \geqslant \eta_{t}(x)\ \forall x \in [1, n]^2 \qquad \text{and}$$
$$\xi^{2n,n}_{t}(x) \geqslant \eta'_{t}(x - (n, 0))\ \forall x \in [n+1, 2n] \times [1, n]$$

for all $t \in [T_i, S_i]$. Therefore to show the proposition it only remains to show that we can choose our coupling so that if $S_i < V_n$, then the above relation holds for t up to and including the smaller of V_n and T_i outside of a set of small probability. We suppose without loss of generality that at S_i, $\eta' = \underline{0}$ and η_{S_i} is not identically zero. Now up until the minimum of the times T_i and V_n, the process η will simply be a contact process generated by its Harris system and so automatically we will have $\eta_t(x) \leqslant \xi^{2n,n}_t(x)$ for $x \in [1, n]^2$ and $t \in [S_i, \min(V_n, T_i)]$. Equally η' is identically vacant on the time interval $[S_i, T_i)$ a.s. Thus it only remains to show that we can couple our processes so that at $\xi^{2n,n}(x) = 1$.

We consider the time interval $[S_i, S_i + n^8]$. We neglect the null set where η dies out at times $S_i + n^8, S_i + 2n^8 \cdots$. Then if η dies out during this interval, then we are free of any coupling constraints and will simply use independent Bernoulli(δ) random variables to generate η' and later η. So we need only treat the case where η is not identically vacant on this interval. Let K_i be the first time in time interval $[S_i, S_i + n^8/2]$ that a site in η_t "tries to give birth" to a site in $[n+1, 2n]$. More formally we write

$$K_i = \inf\left\{ t \in \left[S_i, S_i + \frac{n^8}{2} \right] : \text{for some } k \in [1, n], \right.$$
$$\left. \eta_t(n, k) = 1,\ t \in N_{(n,k),(n+1,k)} \right\}$$

Note that as $\xi^{2n,n}$ dominates η on $[1, n]^2$, it must be the case (for finite K_i) that $\xi^{2n,n}_{K_i}(n+1, k) = 1$ for k in the definition of K_i above. On K_i finite, we consider the contact process on $[n+1, 2n] \times [1, n]$ starting at time K_i with just the site (n + 1,k) occupied. This system is certainly allowed to die out and will by the previous observation be dominated by $\xi^{2n,n}$. We denote this process by $\xi'^{,i}$, so $\xi'^{,i}_{K_i}$ is 1 at site $(n+1, k)$ and zero everywhere else. Now by Corollary 1.1 repeatedly applied we have that $P[K_i = \infty] \leqslant e^{-kn^4}$ for some strictly positive k not depending on n. If in fact $K_i = \infty$ then we will henceforth generate the Bernouilli random variables of the two processes (η, v), (η', v') independently of $\xi^{2n, n}$. In this (unlikely) event the coupling will have broken down. By Corollary 1.2 the conditional probability that $\xi'^{,i}_{S_i+n^8}(x+(n, 0)) = 1$ is at least $c^2 > \delta$. We introduce U_i, a uniform random variable independent of $\xi^{2n, n}$. We define $\eta'_{S_i+n^8}(x) = 1$ if and only if

(A) $\xi'^{,i}_{S_i+n^8}(x+(n, 0)) = 1$ and

(B) $U_i \leqslant \delta / P[\xi'^{,i}_{S_i+n^8}(x+(n, 0)) = 1 \mid K_i(k, n)]$

If this is not so then we repeat. We note that if $\eta'_{S_i+n^8}(x) = 1$, then by design it must be the case that either

(a) $V_n < S_i + n^8$, or

(b) K_i equals infinity, or

(c) $\xi^{2n, 2}_{S_i+n^8}(x+(n, 0)) = 1$.

The probability that for some i, $S_i < V_n$ and $K_i = \infty$ is easily seen to be bounded by $Kn^8 e^{Ln^2} e^{-hn^4} \leqslant e^{-cn^4}$. The result is therefore proven.

Corollary 3.1. Let $\tau^{2n, n}$ be the time for the contact process restricted to $[1, 2n] \times [1, n]$ to die. For all n large, $E[\tau^{2n, n}] \geqslant n^{-9} E[\tau^n]^2$.

Proof. Consider two independent processes (η^n, v^n), $(\eta^{n'}, v^{v'})$ coupled with a contact process restricted to $[1, 2n] \times [1, n]$ as in Proposition 3.1 above. By this proposition we have that

$$P[\tau^{2n, n} < (E[\tau^n])^2/n^{17/2}] < P[V_n < (E[\tau^n])^2/n^{17/2}] + e^{-cn^4}$$

It remains simply to estimate the probability on the right hand side. By Corollary 2.2 it is sufficient to show that if $(\eta^{n, e}, v_n)$, $(\eta^{n, e'}, v'_n)$ are two independent processes started in equilibrium and if $V'_n = \inf\{t : \eta^{n, e} = \eta^{n, e'}_t = \underline{0}\}$, then $P[V_n < (E[\tau^n])^2/n^{17/2}]$ tends to zero as n tends to zero. Let random variable X equal $\int_0^{((E[\tau^n])^2/n^{17/2}) + n^8} I_{\eta^{n, e}_s = \eta^{n, e'}_s = \underline{0}}\, ds$. Then by Lemma 2.1, $E[X] \leqslant (E[\tau^n])^2/n^{17/2} + n^8 (K(n^8/E[\tau^n]))^2 \leqslant n^{8-1/2}$. On the other hand the conditional expectation of X given that V_n is less than $(E[\tau^n]/n^8)^2/n^{1/2}$ is at least $n^8/2$. The result follows.

In the same way we show

Corollary 3.2. Fix M. For n large, $r \geqslant 1$, $nr \leqslant m \leqslant nM$, we have $E[\tau^{m, n(r+1)}] \geqslant n^{-9} E[\tau^{m, nr}]\, E[\tau^{m, n}]$.

We are now ready to prove Theorem 1.1. The proof is very similar to the proof of the existence of a limiting average for sub or super additive sequences. We follow the path we do because we do not wish to prove superadditivity for terms corresponding to the death times for rectangles of wildly differing side length.

Proof of Theorem 1.1. Fix $M \gg 2^7$ and $\varepsilon \ll 1/M$, but otherwise arbitrary. Pick n_0 so large that the conclusions of all the above results hold for our fixed M. We will also require that $72 \log(n) \leqslant n^2 \varepsilon$ for all $n \geqslant n_0$ and that $\varepsilon \log(n_0) \gg 1$.

Let γ be the limsup as n tends to infinity of $\log E[\tau^n]/n^2$. The paper of [C] ensures that γ is in the interval $(0, \infty)$. Let n_1 be the first n greater than n_0 such that $\log(E[\tau^n]) \geqslant n^2(\gamma - \varepsilon/4)$. Now for any n greater than n_0, we have by Corollary 3.1, and then Corollary 3.2 applied with $m = 2n$ and $r = 1$, that $E[\tau^{2n}] \geqslant (E[\tau^n])^4/n^{18}$. That is if $a_n = E[\tau^n]/n^7$, then $a_{2n} \geqslant (a_n)^4$. This means that if $a_n \geqslant e^{n^2(\gamma - \varepsilon)}$, then $\forall i$, $a_{2^i n} \geqslant e^{(n 2^i)^2 (\gamma - \varepsilon)}$. Thus this holds in particular for $n = n_1$.

Now given $k \in [M/2, M]$ apply Corollary 3.2 repeatedly with m and n both equal to n_1 and r equal to $1, 2,..., k-1$, to finally obtain

$$E[\tau^{kn_1, n_1}] = E[\tau^{n_1, kn_1}] \geqslant (E[\tau^{n_1}])^k/(n_1^{9k})$$

Now apply Corollary 3.2 in succession again with $m = kn_1$ and $r = 1, 2,..., k-1$ to obtain

$$E[\tau^{kn_1, kn_1}] \geqslant (E[\tau^{n_1, kn_1}])/(n_1^{9k}) \geqslant (E[\tau^{n_1}])^{k^2}/(n_1^{9k^2+9k})$$

Thus, by our choice of n_0 we have that for every $k \in [M/2, M]$, $\log E[\tau^{kn_1}] \geqslant (\gamma - \varepsilon/2)(kn_1)^2$. Consequently (again using our choice of n_0, we obtain for each $k \in [2, M]$ that $a_{kn_1} \geqslant e^{(\gamma - \varepsilon/2)(n_1 k)^2}/(n_1 k)^7 \geqslant e^{(\gamma - \varepsilon)(n_1 k)^2}$. Thus by preceding paragraph we find that for each $k \in [M/2, M]$ and each positive i, that $E[\tau^{2^i n_1 k}] \geqslant e^{(\gamma - \varepsilon)(2^i k n_1)^2}$. Finally given any $n \geqslant Mn_1$, choose i so that $(M/2)\, n_1 2^i \leqslant n < Mn_1 2^i$, and choose k so that $2^i k n_1 \leqslant n < 2^i (k+1)\, n_1$. Then by monotonicity we have

$$E[\tau^n] \geqslant E[\tau^{(2^i k n_1)^2}] \geqslant e^{(\gamma - \varepsilon)(1 - 2/M)^2 n^2}$$

This proves the result by the arbitrariness of M and ε.

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